Model Checking: the Interval Way

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Model checking: the desired properties of a system are checked against a model of it

- the model is usually a (finite) state-transition system
- system properties are specified by a temporal logic (LTL, CTL, CTL* and the like)

Distinctive features of model checking:

- exhaustive check of all the possible behaviours
- fully automatic process
- a counterexample is produced for a violated property
The Interval Way

Model checking is usually point-based:
- properties express requirements over points (snapshots) of a computation (states of the state-transition system)
- they are specified by means of point-based temporal logics such as LTL, CTL, and CTL*

Interval properties express conditions on computation stretches instead of on computation states

A lot of work has been done on interval temporal logic (ITL) satisfiability checking (a comprehensive survey can be found at: https://users.dimi.uniud.it/~angelo.montanari/Movep2016-part1.pdf).

ITL model checking entered the research agenda only in the last years (Bozzelli, Lomuscio, Michaliszyn, Molinari, Montanari, Murano, Perelli, Peron, Sala)
Outline of the talk

- The model checking problem for interval temporal logics
- Complexity results: the general picture
- Interval vs. point temporal logic model checking: an expressiveness comparison (a short account)
- Interval temporal logic model checking with regular expressions (a short account)
- Ongoing work and future developments
The modeling of the system: Kripke structures

▶ ITL formulas are interpreted over (finite) state-transition systems, whose states are labeled with sets of proposition letters (Kripke structures)

▶ An interval is a trace (finite path) in a Kripke structure

An example of Kripke structure
HS: the modal logic of Allen’s interval relations

Allen’s interval relations: the 13 binary ordering relations between 2 intervals on a linear order. They give rise to corresponding unary modalities over frames where intervals are primitive entities:

- HS features a modality for any Allen ordering relation between pairs of intervals (except for equality)

<table>
<thead>
<tr>
<th>Allen rel.</th>
<th>HS</th>
<th>Definition</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>meets</td>
<td>⟨A⟩</td>
<td>([x, y]\mathcal{R}_A[v, z] \iff y = v)</td>
<td>![Diagram of meets]</td>
</tr>
<tr>
<td>before</td>
<td>⟨L⟩</td>
<td>([x, y]\mathcal{R}_L[v, z] \iff y &lt; v)</td>
<td>![Diagram of before]</td>
</tr>
<tr>
<td>started-by</td>
<td>⟨B⟩</td>
<td>([x, y]\mathcal{R}_B[v, z] \iff x = v \land z &lt; y)</td>
<td>![Diagram of started-by]</td>
</tr>
<tr>
<td>finished-by</td>
<td>⟨E⟩</td>
<td>([x, y]\mathcal{R}_E[v, z] \iff y = z \land x &lt; v)</td>
<td>![Diagram of finished-by]</td>
</tr>
<tr>
<td>contains</td>
<td>⟨D⟩</td>
<td>([x, y]\mathcal{R}_D[v, z] \iff x &lt; v \land z &lt; y)</td>
<td>![Diagram of contains]</td>
</tr>
<tr>
<td>overlaps</td>
<td>⟨O⟩</td>
<td>([x, y]\mathcal{R}_O[v, z] \iff x &lt; v &lt; y &lt; z)</td>
<td>![Diagram of overlaps]</td>
</tr>
</tbody>
</table>

All modalities can be expressed by means of ⟨A⟩, ⟨B⟩, ⟨E⟩, and their transposed modalities only (if point intervals are admitted, ⟨B⟩, ⟨E⟩, and their transposed modalities suffice)
HS semantics and model checking

Truth of a formula $\psi$ over a trace $\rho$ of a Kripke structure $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$ defined by induction on the complexity of $\psi$:

- $\mathcal{K}, \rho \models p$ iff $p \in \bigcap_{w \in \text{states}(\rho)} \mu(w)$, for any letter $p \in \mathcal{AP}$ (homogeneity assumption);
- clauses for negation, disjunction, and conjunction are standard;
- $\mathcal{K}, \rho \models \langle A \rangle \psi$ iff there is a trace $\rho'$ s.t. $\text{lst}(\rho) = \text{fst}(\rho')$ and $\mathcal{K}, \rho' \models \psi$;
- $\mathcal{K}, \rho \models \langle B \rangle \psi$ iff there is a proper prefix $\rho'$ of $\rho$ s.t. $\mathcal{K}, \rho' \models \psi$;
- $\mathcal{K}, \rho \models \langle E \rangle \psi$ iff there is a proper suffix $\rho'$ of $\rho$ s.t. $\mathcal{K}, \rho' \models \psi$;
- the semantic clauses for $\langle A \rangle, \langle B \rangle$, and $\langle E \rangle$ are similar

Model Checking

$\mathcal{K} \models \psi \iff$ for all initial traces $\rho$ of $\mathcal{K}$, it holds that $\mathcal{K}, \rho \models \psi$

Possibly infinitely many traces!
Remark: HS state semantics ($\text{HS}_{st}$)

According to the given semantics, HS modalities allow one to branch both in the past and in the future.
The Kripke structure $\mathcal{K}_{Sched}$ for a simple scheduler
A short account of $K_{Sched}$

$K_{Sched}$ models the behaviour of a scheduler serving 3 processes which are continuously requesting the use of a common resource (it can be easily generalised to an arbitrary number of processes)

Initial state: $v_0$ (no process is served in that state)

In $v_i$ and $\overline{v}_i$ the $i$-th process is served ($p_i$ holds in those states)

The scheduler cannot serve the same process twice in two successive rounds:

- process $i$ is served in state $v_i$, then, after “some time”, a transition $u_i$ from $v_i$ to $\overline{v}_i$ is taken; subsequently, process $i$ cannot be served again immediately, as $v_i$ is not directly reachable from $\overline{v}_i$

- a transition $r_j$, with $j \neq i$, from $\overline{v}_i$ to $v_j$ is then taken and process $j$ is served
Some meaningful properties to be checked over $K_{\text{Sched}}$

Validity of properties over all legal computation intervals can be forced by modality $[E]$ (they are suffixes of at least one initial trace)

**Property 1**: in any computation interval of length at least 4, at least 2 processes are witnessed (YES/no process can be executed twice in a row)

\[
K_{\text{Sched}} \models [E](\langle E \rangle^3 \top \rightarrow (\chi(p_1, p_2) \lor \chi(p_1, p_3) \lor \chi(p_2, p_3)))
\]

where $\chi(p, q) = \langle E \rangle \langle \overline{A} \rangle p \land \langle E \rangle \langle \overline{A} \rangle q$

**Property 2**: in any computation interval of length at least 11, process 3 is executed at least once (NO/if there are at least 3 processes, the scheduler can postpone the execution of one of them ad libitum—starvation)

\[
K_{\text{Sched}} \not\models [E](\langle E \rangle^10 \top \rightarrow \langle E \rangle \langle \overline{A} \rangle p_3)
\]

**Property 3**: in any computation interval of length at least 6, all processes are witnessed (NO/the scheduler should be forced to execute them in a strictly periodic manner, which is not the case)

\[
K_{\text{Sched}} \not\models [E](\langle E \rangle^5 \rightarrow (\langle E \rangle \langle \overline{A} \rangle p_1 \land \langle E \rangle \langle \overline{A} \rangle p_2 \land \langle E \rangle \langle \overline{A} \rangle p_3))
\]
Model checking: the key notion of $BE_k$-descriptor

- The BE-nesting depth of an HS formula $\psi$ ($\text{Nest}_{BE}(\psi)$) is the maximum degree of nesting of modalities $B$ and $E$ in $\psi$.
- Two traces $\rho$ and $\rho'$ of a Kripke structure $\mathcal{K}$ are $k$-equivalent if and only if $\mathcal{K}, \rho \models \psi$ iff $\mathcal{K}, \rho' \models \psi$ for all HS-formulas $\psi$ with $\text{Nest}_{BE}(\psi) \leq k$. 
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For any given $k$, we provide a suitable tree representation for a trace, called a $BE_k$-descriptor.

The $BE_k$-descriptor for a trace $\rho = v_0 v_1 .. v_{m-1} v_m$, denoted $BE_k(\rho)$, has the following structure:

$$ (v_0, \{v_1, .., v_{m-1}\}, v_m) \leftarrow \text{descriptor element}$$

**Remark:** the descriptor does not feature sibling isomorphic subtrees.
An example of a $BE_2$-descriptor

The $BE_2$-descriptor for the trace $\rho = v_0v_1v_4^4v_1$ - point intervals are excluded (for the sake of readability, only the subtrees for prefixes are displayed)
An example of a $BE_2$-descriptor

The $BE_2$-descriptor for the trace $\rho = v_0 v_1 v_0^4 v_1$ - point intervals are excluded (for the sake of readability, only the subtrees for prefixes are displayed)

Remark: the subtree to the left is associated with both prefixes $v_0 v_1 v_0^3$ and $v_0 v_1 v_0^4$ (no sibling isomorphic subtrees in the descriptor)
Decidability of model checking for full HS

**FACT 1:** For any Kripke structure $\mathcal{K}$ and any BE-nesting depth $k \geq 0$, the number of different $BE_k$-descriptors is finite (and thus at least one descriptor has to be associated with infinitely many traces)
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**Theorem**

The model checking problem for full HS on finite Kripke structures is **decidable** (with a non-elementary algorithm).

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What about lower bounds?
The logic BE

Theorem

The model checking problem for BE, over finite Kripke structures, is EXPSPACE-hard


Proof: a polynomial-time reduction from a domino-tiling problem for grids with rows of single exponential length

- for an instance $I$ of the problem, we build a Kripke structure $K_I$ and a BE formula $\varphi_I$ in polynomial time
- there is an initial trace of $K_I$ satisfying $\varphi_I$ iff there is a tiling of $I$
- $K_I \models \neg \varphi_I$ iff there exists no tiling of $I$
BE hardness: encoding of the domino-tiling problem

Instance of the tiling problem: \((C, \Delta, n, d_{\text{init}}, d_{\text{final}})\), with \(C\) a finite set of colors and \(\Delta \subseteq C \times C \times C \times C\) a set of tuples \((c_B, c_L, c_T, c_R)\)

\[
\begin{array}{cccccccc}
  d_0^k & d_1^k & d_2^k & & & & & d_{2^n-2}^k & d_{2^n-1}^k \\
  & & d_i^{i+1} & & & & & & d_{F_{\text{in}}} \\
  d_{i-1}^i & d_i^i & d_{i+1}^i & & & & & & \\
  d_i^{i-1} & & & & & & & & \\
  d_0^0 & d_1^0 & d_2^0 & & & d_{2^n-2}^0 & d_{2^n-1}^0 \\
\end{array}
\]

String (interval) encoding of the problem

row 0: \[
\begin{array}{cccccccc}
  d_0^0 & 0 \ldots 00 & d_1^0 & 1 \ldots 00 & \ldots & d_{2^n-2}^0 & 1 \ldots 11 & d_{2^n-1}^0 & 1 \ldots 11 \\
\end{array}
\]

row 1: \[
\begin{array}{cccccccc}
  d_0^1 & 0 \ldots 00 & d_1^1 & 1 \ldots 00 & \ldots & d_{2^n-2}^1 & 1 \ldots 11 & d_{2^n-1}^1 & 1 \ldots 11 \\
\end{array}
\]

Model Checking: the Interval Way

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The complexity picture

- A\overline{B}\overline{B}, A\overline{A}\overline{B}\overline{E}, A\overline{A}E\overline{B} - EXPSPACE
  - PSPACE-hard

- A\overline{A}B\overline{B} - PSPACE-complete
  - hardness

- A\overline{A}B\overline{E} - PSPACE-complete
  - hardness

- A\overline{B}E - PSPACE-complete
  - hardness

- AB - PNP-complete
  - upper-bound

- AE - PNP-complete
  - upper-bound

- A\overline{A} - PNP[O(log^2 n)]
  - \( p_{NP}[O(log^2 n)] \)-hard

- A\overline{A} - PNP[O(log n)]
  - hardness

- A, \overline{A} - PNP[O(log^2 n)]
  - \( p_{NP}[O(log^2 n)] \)-hard

- A\overline{B}, A\overline{E} - PNP[O(log^2 n)]
  - hardness

- \overline{A}B, AE - PNP[O(log n)]
  - \( p_{NP}[O(log n)] \)-hard

- E - coNP-complete
  - hardness

- B - coNP-complete
  - hardness

- Prop - coNP-complete
  - hardness

- full HS - nonELEMENTARY
  - EXPSPACE-hard

- BE - nonELEMENTARY
  - EXPSPACE-hard

- A\overline{B}\overline{B}, A\overline{A}\overline{B}\overline{E}, A\overline{A}E\overline{B} - EXPSPACE
  - PSPACE-hard

- A\overline{A}B\overline{B} - PSPACE-complete
  - hardness

- A\overline{A}B\overline{E} - PSPACE-complete
  - hardness

- A\overline{B}E - PSPACE-complete
  - hardness

- AB - PNP-complete
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- AE - PNP-complete
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- A\overline{A} - PNP[O(log^2 n)]
  - \( p_{NP}[O(log^2 n)] \)-hard

- A\overline{A} - PNP[O(log n)]
  - hardness

- A, \overline{A} - PNP[O(log^2 n)]
  - \( p_{NP}[O(log^2 n)] \)-hard

- A\overline{B}, A\overline{E} - PNP[O(log^2 n)]
  - hardness

- \overline{A}B, AE - PNP[O(log n)]
  - \( p_{NP}[O(log n)] \)-hard

- E - coNP-complete
  - hardness

- B - coNP-complete
  - hardness

- Prop - coNP-complete
  - hardness
Three main gaps to fill

There are three main gaps to fill:

- full HS and BE are in between nonELEMENTARY and EXPSPACE

- AABBE, AÆEÆE, ABÆE, AEÆE, ABÆE, and AÆÆE are in between EXPSPACE and PSPACE

- A, À, ÀÀ, ÀB, and AE are in between $P^{NP}[O(\log^2 n)]$ and $P^{NP}[O(\log n)]$

The first gap is definitely the most significant one
Point vs. interval temporal logic model checking

**Question**: is there any advantage in replacing points by intervals as the primary temporal entities, or is it just a matter of taste?

In order to compare the **expressiveness** of HS in model checking with those of LTL, CTL, and CTL*, we consider three semantic variants of HS:

- HS with state-based semantics (the original one)
- HS with computation-tree-based semantics
- HS with trace-based semantics

These variants are compared with the above-mentioned standard temporal logics and among themselves

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Branching semantic variant of HS

State-based semantics of HS (HS$_{st}$):

- both the future and the past are branching

Linear-past semantic variant of HS

Computation-tree-based semantics of HS ($H_{S_{ct}}$):

- the future is branching
- the past is linear, finite and cumulative
- similar to CTL* + linear past

Linear semantic variant of HS

Trace-based semantics of HS ($\text{HS}_{\text{lin}}$):
- neither the past not the future is branching
- similar to LTL + past
The expressiveness picture

\[ \begin{align*}
& \vdash \text{finitary } \text{CTL}^* \quad \prec \quad \text{CTL}^* \\
\text{HS}_{lp} & \equiv \text{finitary } \text{CTL}^*
\end{align*} \]
Can we relaxe the homogeneity assumption? The addition of regular expressions:

\[ r ::= \varepsilon \mid \phi \mid r \cup r \mid r \cdot r \mid r^* \]

where \( \phi \) is a Boolean (propositional) formula over \( \mathcal{AP} \).

Examples:

\( r_1 = (p \land s) \cdot s^* \cdot (p \land s) \)

\( r_2 = (\neg p)^* \)

\( \rho = v_0 v_1 v_0 v_1 v_1 \)

\( \mu(\rho) = \{p, s\}\{q, s\}\{p, s\}\{q, s\}\{q, s\} \)

\( \rho' = v_0 v_1 v_1 v_1 v_0 \)

\( \mu(\rho') = \{p, s\}\{q, s\}\{q, s\}\{q, s\}\{q, s\}\{p, s\} \)

\[ \mu(\rho) \not\in \mathcal{L}(r_1), \text{ but } \mu(\rho') \in \mathcal{L}(r_1) \]

\[ \mu(\rho) \not\in \mathcal{L}(r_2) \text{ and } \mu(\rho') \not\in \mathcal{L}(r_2) \]
ITL model checking with regular expressions

In the definition of the truth of a formula \( \psi \) over a trace \( \rho \) of a Kripke structure \( \mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0) \), we replace the clause for propositional letters by a clause for regular expressions:

\[ \mathcal{K}, \rho \models r \iff \mu(\rho) \in L(r) \]

Homogeneity can be recovered as a special case. To force it, all regular expressions in the formula must be of the form:

\[ \rho \cdot (\rho)^* \]

**Solution**: given \( \mathcal{K} \) and an HS formula \( \varphi \) over \( \mathcal{AP} \), we build an NFA over \( \mathcal{K} \) accepting the set of traces \( \rho \) such that \( \mathcal{K}, \rho \models \varphi \).

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Ongoing work: to determine the exact complexity of the satisfiability / model checking problems for BE over finite linear orders, under the homogeneity assumption (the three semantic variants of HS coincide over BE)

We know that the satisfiability/model checking problems for D over finite linear orders, under the homogeneity assumption, are PSPACE-complete (we exploit a spatial encoding of the models for D and a suitable contraction technique)

L. Bozzelli, A. Molinari, A. Montanari, A. Peron, and P. Sala, Satisfiability and Model Checking for the Logic of Sub-Intervals under the Homogeneity Assumption, Proc. of the 44th International Colloquium on Automata, Languages, and Programming (ICALP), LIPIcs 80, July 2017, pp. 120:1–120:14

There is no a natural way to generalize the solution for D to BE
Ongoing work and future developments - 2

Ongoing work: we are looking for possible replacements of Kripke structures by more expressive system models

▶ inherently interval-based models, that allows one to directly describe systems on the basis of their interval behavior/properties, such as, e.g., those involving actions with duration, accomplishments, or temporal aggregations (no restriction on the evaluation of proposition letters)

▶ timeline-based (planning) systems: a set of timelines (transition functions) plus a set of synchronization rules

▶ visibly pushdown systems, that can encode recursive programs and infinite state systems

A different direction: model checking a single interval model (for temporal dataset evaluation)