Extending $\mathcal{ALC}$ with the power-set construct

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Aim of the talk

$\mathcal{ALC}$ and Ω

The description logic $\mathcal{ALC}^{\Omega}$

A set-theoretic translation of $\mathcal{ALC}^{\Omega}$

Aim of the talk

We explore the relationships between Description Logics and Set Theory.

- On the set-theoretic side, we consider a very rudimentary axiomatic set theory Ω, consisting of only four axioms characterizing binary union, set difference, inclusion, and the power-set.
- We consider *an extension of the description logic $\mathcal{ALC}$*, $\mathcal{ALC}^{\Omega}$ [ICTCS 2018], in which *concepts* are naturally interpreted *as sets* living in Ω-models: *membership between concepts* and *power-set construct* to add metamodeling capabilities.

  In previous work we defined a *polynomial translation of* $\mathcal{ALC}^{\Omega}$ in the DL $\mathcal{ALCOI}$ (showing that concept satisfiability in $\mathcal{ALC}^{\Omega}$ is ExpTime-complete)
- In this paper we develop a *set-theoretic translation* of the description logic $\mathcal{ALC}^{\Omega}$ in the set theory Ω

Motivations: metamodeling capabilities

The idea of enhancing the language of description logics with statements of the form $C \in D$, with C and D concepts is not new: similar assertions are allowed in OWL-Full.

Example [Welty1994,Motik05]

One can represent the fact that *eagles are in the red list of endangered species*, by the axiom

$$Eagle \in RedListSpecies$$

and that *Harry is an eagle*, by the assertion

$$harry \in Eagle.$$

The *power-set concept*, $\texttt{Pow}(C)$, allows to capture in a natural way the interactions between concepts and metaconcepts.

$$RedListSpecies \sqsubseteq \texttt{Pow}(CannotHunt),$$

means that: *"all the instances of the species in the Red List are not allowed to be hunted"*

The theory Ω

- The first-order theory Ω consists of the four axioms

$$
\begin{aligned}
x \in y \cup z &\leftrightarrow x \in y \vee x \in z; \\
x \in y \backslash z &\leftrightarrow x \in y \wedge x \notin z; \\
x \subseteq y &\leftrightarrow \forall z (z \in x \to z \in y); \\
x \in Pow(y) &\leftrightarrow x \subseteq y.
\end{aligned}
$$

- In any Ω-model *everything* is supposed to be a set, and circular definition of sets are not forbidden
- no *extensionality* axiom: there are Ω-models in which different sets have equal collection of elements.
- The most natural Ω-model is the collection of well-founded sets $\mathrm{HF} = \mathrm{HF}^0 = \bigcup_{n \in \mathbb{N}} \mathrm{HF}_n$, where: $\mathrm{HF}_0 = \emptyset$ and $\mathrm{HF}_{n+1} = Pow(\mathrm{HF}_n)$.

The description logic $\mathcal{ALC}^{\Omega}$ [Ictcs 2018]

The set of $\mathcal{ALC}^{\Omega}$ *concepts* are defined inductively as follows:

- $A \in N_C$, $\top$ and $\bot$ are $\mathcal{ALC}^{\Omega}$ *concepts*;
- if C, D are $\mathcal{ALC}^{\Omega}$ concepts and $R \in N_R$, then the following are $\mathcal{ALC}^{\Omega}$ *concepts*:

$$C \sqcap D,\ C \sqcup D,\ \neg C,\ C \backslash D,\ \mathtt{Pow}(C),\ \forall R.C,\ \exists R.C$$

New membership axioms: $C \in D$ and $(C, D) \in R$ besides the standard assertions $D(a)$ and $R(c, d)$

General concepts (and not only concept names) can be instances of other concepts, e.g., *polar bears are in the red list of endangered species*,

$$Polar \sqcap Bear \in RedListSpecies$$

and *polar bears are more endangered than eagles* by the role membership axiom

$$(Polar \sqcap Bear, Eagle) \in moreEndangered$$

Semantics of $\mathcal{ALC}^\Omega$

An *interpretation* for $\mathcal{ALC}^\Omega$ is a pair $I = \langle \Delta, \cdot^I \rangle$ *over a set of atoms* $\mathbb{A}$ where:

- the non-empty domain Δ is a transitive set (i.e., $(\forall y \in \Delta)(y \subseteq \Delta)$) chosen *in the universe* $\mathcal{U}$ *of a model of* Ω over the atoms in $\mathbb{A}$
- the extension function $\cdot^I$ maps each concept name $A \in N_C$ to an element $A^I \subseteq \Delta$; each role name $R \in N_R$ to a binary relation $R^I \subseteq \Delta \times \Delta$; and each individual name $a \in N_I$ to an element $a^I \in \mathbb{A} \cap \Delta$. The function $\cdot^I$ is extended to complex concepts of $\mathcal{ALC}^\Omega$ as follows:

$$\top^I = \Delta \qquad \bot^I = \emptyset \qquad (\neg C)^I = \Delta \backslash C^I$$

$$(C \backslash D)^I = (C^I \backslash D^I) \qquad (\mathcal{Pow}(C))^I = Pow(C^I) \cap \Delta$$

$$(C \sqcap D)^I = C^I \cap D^I \qquad (C \sqcup D)^I = C^I \cup D^I$$

$$(\forall R.C)^I = \{x \in \Delta \mid \forall y((x, y) \in R^I \rightarrow y \in C^I)\}$$

$$(\exists R.C)^I = \{x \in \Delta \mid \exists y((x, y) \in R^I \wedge \ y \in C^I)\}$$

Semantics of $\mathcal{ALC}^{\Omega}$

Observe that

- Δ is not guaranteed to be closed under union, intersection, etc., the interpretation C^I of a concept C *is a set in* $\mathcal{U}$*, but not necessarily an element of* Δ.
- However, $C^I \subseteq \Delta$, as the interpretation of the power-set concept $(\mathtt{Pow}(C))^I = (\mathtt{Pow}(C^I)) \cap \Delta$ is the *portion of the (set-theoretic) power-set visible in* Δ.

Example

Let $K = (\mathcal{T}, \mathcal{A})$ be the set of inclusions and assertions:

(1) $\mathit{ReadingGroup} \sqsubseteq \mathtt{Pow}(\mathit{Person})$
(2) $\mathit{Meeting} \sqsubseteq \mathtt{Pow}(\mathit{ReadingGroup})$
(3) $\mathit{Meeting} \sqsubseteq \mathtt{Pow}(\exists \mathit{has\_leader}.\mathit{Person})$
(4) $\mathit{SummerMeeting} \sqsubseteq \mathtt{Pow}(\exists \mathit{has\_paid}.\mathit{Fee})$

$\mathit{HistoryGroup}, \mathit{FantasyGroup}, \mathit{ScienceGroup} \in \mathit{ReadingGroup};$
$\mathit{SummerMeeting}, \mathit{WinterMeeting} \in \mathit{Meeting};$
$\mathit{ScienceGroup}, \mathit{FantasyGroup} \in \mathit{SummerMeeting};$
$\mathit{bob} \in \mathit{FantasyGroup};\ \mathit{alice}, \mathit{bob} \in \mathit{ScienceGroup};\ \mathit{carl} \in \mathit{History}$

Each reading group is a set of persons (1). The history, fantasy and science groups are reading groups. Each meeting is a set of reading groups (2). The *SummerMeeting* and the *WinterMeeting* are meetings. Both the Science group and the Fantasy group participate to the *SummerMeeting*. Each reading group in a meeting has a leader, who is a person (3). All participants to the *SummerMeeting* have paid the fee (4).

Polynomial encoding of $\mathcal{ALC}^{\Omega}$ into $\mathcal{ALCOI}$

- each concept C of $\mathcal{ALC}^{\Omega}$ is translated to a concept C^T of $\mathcal{ALCOI}$ by replacing all occurrences of the power-set concept $\texttt{Pow}(C)$ with $\forall e.C$;
- a new *individual name* e_C is added, *for each concept name* C occurring on the left hand side of a membership axiom $C \in D$, which is translated to an assertion $D^T(e_C)$ (similarly for role membership axioms);
- the role e relates e_C with all the instances of concept C, by axiom

$$C^T \equiv \exists e^{-}.\{e_C\}$$

- for each (standard) individual name $a \in N_I$, the assertion $(\neg \exists e.\top)(a)$ is added.

Soundness and completeness of the polynomial translation in $\mathcal{ALCOI}$ provide, besides decidability, an EXPTIME upper bound for satisfiability in $\mathcal{ALC}^{\Omega}$.

Example: Translation in $\mathcal{ALCOI}$

Let $K = (\mathcal{T}, \mathcal{A})$ be the knowledge base with TBox $\mathcal{T}$

$$RedListSpecies \sqsubseteq \mathtt{Pow}(CannotHunt)$$

and ABox $\mathcal{A}$

$$Eagle(harry), Eagle \in RedListSpecies,$$
$$Polar \sqcap Bear \in RedListSpecies$$

K is translated into $K^T = (\mathcal{T}^T, \mathcal{A})^T$ with TBox $\mathcal{T}^T$:

$$RedListSpecies \sqsubseteq \forall e.CannotHunt,$$
$$Eagle \equiv \exists e^-.\{e_{Eagle}\}$$
$$Polar \sqcap Bear \equiv \exists e^-.\{e_{Polar \sqcap Bear}\}$$

and ABox $\mathcal{A}^T$:

$$Eagle(harry), \quad RedListSpecies(e_{Eagle}),$$
$$(\neg \exists e.\top)(harry), \quad RedListSpecies(e_{Polar \sqcap Bear})$$

Set-theoretic translation of $\mathcal{ALC}^{\Omega}$ in the set theory Ω

- Our translation of $\mathcal{ALC}^{\Omega}$ into Ω, exploits the *correspondence between membership* $\in$ and the *accessibility relation of a normal modality R* explored in [D'Agostino et al.1995].
- Step by step
  - A set-theoretic translation of $\mathcal{ALC}$ based on *Schild's correspondence with polymodal logics*.
  - A translation of the fragment $\mathcal{LC}^{\Omega}$ of $\mathcal{ALC}^{\Omega}$ without roles and individual names.
  - An encoding of $\mathcal{ALC}^{\Omega}$ into the fragment $\mathcal{LC}^{\Omega}$

Set-theoretic translation of $\mathcal{LC}^\Omega$ in the set theory Ω

- $A \in N_C$, $\top$ and $\bot$ are $\mathcal{LC}^\Omega$ *concepts*;
- if C, D are $\mathcal{LC}^\Omega$ concepts, the following are $\mathcal{LC}^\Omega$ *concepts*:

$$C \sqcap D, \quad C \sqcup D, \quad \neg C, \quad C \backslash D, \quad \mathtt{Pow}(C)$$

$$\top^S = x \qquad \bot^S = \emptyset$$

$$A_i^S = x_i \text{ , for } A_i \text{ in } K \qquad (\neg C)^S = x \backslash C^S$$

$$(C \sqcap D)^S = C^S \cap D^S \qquad (C \sqcup D)^S = C^S \cup D^S$$

$$(C \backslash D)^S = C^S \backslash D^S \qquad (\mathtt{Pow}(C))^S = Pow(C^S)$$

$C_1 \sqsubseteq C_2$ *in TBox* *is translated:* $C_1^S \cap x \subseteq C_2^S$

$C_1 \in C_2$ *in ABox* *is translated:* $C_1^S \in C_2^S$

$K \models_{\mathcal{LC}} C \sqsubseteq D$ *if and only if*

$\Omega \vdash \forall x(Trans(x) \rightarrow$

$\forall x_1, \ldots, \forall x_n(\bigwedge ABox_{\mathcal{A}} \wedge \bigwedge TBox_{\mathcal{T}} \rightarrow C^S \cap x \subseteq D^S))$

Set-theoretic translation of $\mathcal{ALC}$ in the set theory Ω

$$\top^S = x; \qquad \bot^S = \emptyset;$$
$$A_i^S = x_i \text{ , for } A_i \text{ in } K; \qquad (\neg C)^S = x \backslash C^S;$$
$$(C \sqcap D)^S = C^S \cap D^S; \qquad (C \sqcup D)^S = C^S \cup D^S;$$
$$(\forall R_i.C)^S = Pow(((x \cup y_1 \cup \ldots \cup y_k) \backslash y_i) \cup Pow(C^S))$$

A set U_i (represented by the variable y_i) is used to translate role R_i: $(v, v') \in R_i^I$ iff there is some $u_i \in U_i$ such that $v' \in u_i \in v$.

$$C_1 \sqsubseteq C_2 \quad \textit{is translated:} \quad C_1^S \cap x \subseteq C_2^S$$

$$K \models_{\mathcal{ALC}} C \sqsubseteq D \ \textit{ if and only if}$$
$$\Omega \vdash \forall x \forall y_1 \ldots \forall y_k (Trans^2(x) \wedge Axiom_H(x, y_1, \ldots, y_k)$$
$$\rightarrow \forall x_1, \ldots, \forall x_n (\bigwedge TBox_{\mathcal{T}} \rightarrow C^S \cap x \subseteq D^S))$$

This set-theoretic translation of $\mathcal{ALC}$ is based on *Schild's correspondence result* [Schild91]and on the set-theoretic *translation for normal polymodal logics* in [DAgostino1995].

Set-theoretic translation of $\mathcal{ALC}^\Omega$ in $\mathcal{LC}^\Omega$

Given an $\mathcal{ALC}^\Omega$ knowledge base K, we define the encoding K^E of K in $\mathcal{LC}^\Omega$:

$C^E \sqcap \neg(U_1 \sqcup \ldots \sqcup U_k) \sqsubseteq D^E$, $\quad$ $C \sqsubseteq D \in K$

$C^E \in D^E$ $\quad$ $C \in D$ in K;

$a_i^E \in C^E$ $\quad$ $C(a_i)$ in K;

$a_j^E \in F^i_{h,j} \in a_h^E$ and $F^i_{h,j} \in U_i$ $\quad$ $R_i(a_h, a_j)$;

$C_j^E \in G^i_{C_h,C_j} \in C_h^E$ and $G^i_{C_h,C_j} \in U_i$ $\quad$ $R_i(C_h, C_j)$.

The following additional axioms are also needed in K^E:

$A_i \sqsubseteq \neg(U_1 \sqcup \ldots \sqcup U_k)$, one for each concept name A_i in K;

$B_i \in \neg(U_1 \sqcup \ldots \sqcup U_k)$, one for each individual name a_i in K;

$C^E \in \neg(U_1 \sqcup \ldots \sqcup U_k)$, one for each $C \in D$ in K

$\neg(U_1 \sqcup \ldots \sqcup U_k) \sqsubseteq \mathtt{Pow}(\neg(U_1 \sqcup \ldots \sqcup U_k) \sqcup \mathtt{Pow}(\neg(U_1 \sqcup \ldots \sqcup U_k)))$

Set-theoretic translation of $\mathcal{ALC}^{\Omega}$ in the set theory Ω

$\mathcal{LC}^{\Omega}$ has the same expressive power as $\mathcal{ALC}^{\Omega}$:
universal and existential restrictions of the language $\mathcal{ALC}^{\Omega}$, as well as all assertions, can be encoded into $\mathcal{LC}^{\Omega}$.

The encoding, together with the set-theoretic translation of $\mathcal{LC}^{\Omega}$ given in the previous section, determines a set-theoretic translation for $\mathcal{ALC}^{\Omega}$,

$$\top^* = x \qquad \bot^* = \emptyset$$
$$A_i^* = x_i \qquad (\neg C)^* = x \backslash C^*$$
$$(C \sqcap D)^* = C^* \cap D^* \qquad (C \sqcup D)^* = C^* \cup D^*$$
$$(\forall R_i.C)^* = Pow(((x \cup y_1 \cup \ldots \cup y_k) \backslash y_i) \cup Pow(C^*))$$
$$\mathtt{Pow}(C)^* = Pow((y_1 \cup \ldots \cup y_k \cup C^*)$$

$C_1 \sqsubseteq C_2$ *is translated:* $C_1^S \cap (x \backslash (y_1 \cup \ldots \cup y_k)) \subseteq C_2^S$
$C_1 \in C_2$ *is translated:* $C_1^S \in C_2^S \cap (x \backslash (y_1 \cup \ldots \cup y_k))$

Discussion and future work

- The complementary problem to subsumption corresponds to the satisfiability of a formula in the existential fragment of Ω.
- The decidability of subsumption $\mathcal{ALC}^{\Omega}$ comes from the translation into $\mathcal{ALCOI}$.
- The problem of deciding the *satisfiability of an existential formula* of a set theory with power-set is decidable under the *extensionality* and *well-foundedness* assumptions [Cantone et al, 1985]
- Possibility of defining, through a set-theoretic translation as the one above, a *variant of $\mathcal{ALC}^{\Omega}$ with well-founded sets*.
  - is there a translation of $\mathcal{ALC}^{\Omega}$ with extensionality and well-foundedness in DLs? Which complexity?
- Can the set-theoretic translation of $\mathcal{ALC}^{\Omega}$ be extended to expressive DLs? As the finite model property does not hold for them, alternative proof techniques will be needed.

Related work

- Motik [ISWC, 2005] proved that *metamodelling in $\mathcal{ALC}$-Full is undecidable* due to free mixing of logical and metalogical symbols.
  $\Rightarrow$ two decidable semantics, a contextual π semantics and a Hilog ν-semantics.
- [DeGiacomo et al.,11] *Hi*($\mathcal{SHIQ}$) and [Homola et al., 14] $\mathcal{TH}(\mathcal{SROIQ})$ employ an Hilog-style semantics:
  - [Motik,05] and [DeGiacomo et al.,11]: *untyped higher-order* languages (a concept can be an instance of itself); polynomial translation of *Hi*($\mathcal{SHIQ}$) into $\mathcal{SHIQ}$
  - [Homola et al., 14]: *typed higher-order* extension of $\mathcal{SROIQ}$ allowing for a hierarchy of concepts; translation with axioms $A' \equiv \exists instanceOf.\{c_{A'}\}$, for each atomic concept A'

Related work (contd.)

- Kubincova et al. (2015) propose a Hylog-style semantics by dropping the ordering requirement in [Homola et al., 14] and using *the instanceOf role in axioms as any other role*.
- Pan and Horrocks (2005) and Motz et al. (2015) define extensions of OWL DL and of $\mathcal{SHIQ}$ (respectively), based on semantics interpreting *concepts as well-founded sets*.
- Gu (2016) introduces the language *Hi(Horn-SROIQ)*, an extension of Horn-SROIQ which allows classes and roles to be used as individuals based on the ν-semantics. Reduction to Horn-SROIQ.

Thank you!!!