Does every recursively enumerable set admit a finite-fold Diophantine representation?

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Presentation Spirit and Outline

This is a review paper unifying many relevant works in the field.

We will investigate formulas like

$$\exists\, a,\ d,\ \ell,\ s, x,\ h \Bigg[\ (c-1)^2 + n = 0\ \vee (n \geqslant 1\ \&\ c + b = 0)\ \vee$$
$$\Big(n \geqslant 1\ \ \&\ \ b \geqslant 1\ \ \&\ \ \mathcal{J}(a,d)\ \ \&\ \ d > \ell\ \ \&\ \ a > b + n \qquad \&$$
$$\ell^2 = (a^2-1)\left[n + (a-1)\,s\right]^2 + 1\ \ \&\ \ Q(b+n-2,h) = x^2\ \ \&$$
$$2\,a\,b - b^2 - 1 \geqslant \left[(b+n+1)\,x\right] \max\left(c+1\right) \qquad \&$$
$$2\,a\,b - b^2 - 1\ \ |\,\ell - \ \left(a-b\right)\left[(a-1)\,s + n\right]\ -\ c\ \Big)\ \Bigg]$$

Presentation Spirit and Outline

Wait, don't run away . . . I was joking

Technical details will be avoided and we will

- gently introduce the problem
- motivate our interest
- suggest a route towards the answer

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If your are interested on the details, have a look at the paper

Definition (Existentially Definable Relations)

$\mathcal{R} \subseteq \mathbb{N}^n$ is ***existentially definable in terms of*** $\mathcal{J}(\bullet, \ldots, \bullet)$ if

$$\mathcal{R}(a_1, \ldots, a_n) \iff (\exists x_1 \cdots \exists x_m)\, \varphi(a_1, \ldots, a_n, x_1, \ldots, x_m)$$

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When $\mathcal{J}$ is absent, $\mathcal{R}$ is also called ***Diophantine***.

Some examples

- $a = b + x + 1$
- $a = (x + 2) \cdot (y + 2) \lor a + x = 1$
- $b = x \cdot a + y + 1 \;\&\; b = y + 1 + z + 1$
- $a^a = 1 \;\&\; x^x = a + 1$

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- $a = b + x + 1$
  existentially defines $a > b$
- $a = (x+2)\cdot(y+2) \vee a + x = 1$
  existentially defines a **is not prime**
- $b = x \cdot a + y + 1 \;\&\; b = y + 1 + z + 1$
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Many useful Diophantine constructs can be added as done for
$\bullet > \bullet$, $\bullet \nmid \bullet$, and $\bullet$ **is not prime**

Theorem (DPR theorem [DPR61])

Every r.e. set is existentially definable in terms of exponentiation

Two important theorems

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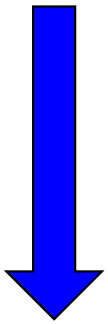

"After the DPR-theorem was proved in 1961, in order to establish the existence of Diophantine representations for *every* effectively enumerable set it was sufficient to find a Diophantine representation for *one particular* set of triples

$$\{\langle a, b, c\rangle \mid a = b^c\}. \qquad (12)$$

"

[Mat10, p. 748]

Theorem (Matiyasevich's theorem (MRDP) [Mat74])

Every r.e. set is existentially definable

Definition (Single-fold existential definitions)

An existential definition

$$\exists \vec{x} \quad \varphi(\vec{a}, \vec{x})$$

(as above) is ***single-fold*** if

$$\forall \vec{a} \forall \vec{x} \forall \vec{y} \left[\quad \varphi(\vec{a}, \vec{x}) \ \& \ \varphi(\vec{a}, \vec{y}) \quad \Longrightarrow \quad \vec{x} = \vec{y} \quad \right]$$

(i.e., $\varphi(a_1, \ldots, a_n , x_1, \ldots, x_m)$ never has multiple solutions).

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Finite-fold existential definitions

The definition of ***finite-fold***-ness is akin:

To each $\vec{a}$ there must correspond a *finite* number of solutions.

A Significant Improvement to DPR

Theorem (An improvement to DPR [Mat74])

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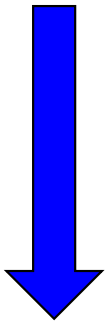

"Today we are in a similar position with respect to single-fold (and finite-fold) Diophantine representations" (...) "it would be sufficient to find a single-fold (or, respectively, finite-fold) Diophantine representation for the same set of triples" i.e., $\{\langle a, b, c\rangle \mid a = b^c\}$. [Mat10, p. 748]

Aimed result

Every r.e. set admits an existential *single/finite-fold* definition.

A Motivating Application for Finite Foldness

Let $P(a, \vec{x})$ be Diophantine, let $\mathcal{M}$ be the set:

$$a \in \mathcal{M} \iff \exists \vec{x} \, \{P(a, \vec{x}) = 0\},$$

and let $\mathcal{M}_n$ be an initial fragment of $\mathcal{M}$, i.e.,

$$\mathcal{M}_n = \mathcal{M} \cap \{k \mid k \leqslant n\}$$

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- trivially, n bits: 1 iff $k \in \mathcal{M}$
- if $\mathcal{M}$ is recursive, $\log n$ bits
- if $\mathcal{M}$ is r.e., $2 \log n$ bits to transmit both n and k. $0 \in \mathcal{M} \| \ldots \| n \in \mathcal{M}$ is evaluated until we get k positive answers

A Motivating Application for Finite Foldness

Chaitin [Cha87] built two special Diophantine formulas and a $\mathcal{M}$

$$a \in \mathcal{M} \iff \exists^{\infty} \vec{x} \; \{E_L(a, \vec{x}) \; = \; E_R(a, \vec{x})\}, \tag{9}$$

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such that

> ... *"whatever so-called prefix-free compression algorithm is used, n bits (up to an additive constant) are required for representing the initial fragment (9) of $\mathcal{M}$"*
>
> [Mat10, p. 747]

Introducing Details and...

If there exists finite-fold Diophantine $\mathcal{D} \subseteq \mathbb{N} \times \mathbb{N}$ statifying

There exist integers $\alpha > 1$, $\beta \geqslant 0$, $\gamma \geqslant 0$, $\delta > 0$ such that to each $w \in \mathbb{N}$ other than 0 there correspond p, q such that $\mathcal{D}(p, q)$, $p < \gamma\, w^{\beta}$, and $q > \delta\, \alpha^{w}$ hold.

then exponentiation is finite-fold Diophantine [Mat10]

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☞ 4 candidates for $\mathcal{D}$ proposed at CILC one year ago

Thank you for the attention!


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A "sibling" of the halting problem

One can find a concrete polynomial

$$H \in \mathbb{Z}[a, x_0, x_1, \ldots, x_\kappa, y, w]$$

such that

1. to each $\boldsymbol{a} \in \mathbb{N}$, there corresponds at most one tuple $\langle \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\kappa, \boldsymbol{u} \rangle \in \mathbb{N}^{\kappa+2}$ s.t. $H(\boldsymbol{a}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_\kappa, \boldsymbol{u}, 2^{\boldsymbol{u}}) > 0$;

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2. to any monadic totally computable function $\mathcal{C}$, there correspond tuples $\langle a, v_0, v_1, \ldots, v_\kappa, u\rangle \in \mathbb{N}^{\kappa+3}$ s.t.

$$\begin{aligned} H(a, v_0, v_1, \ldots, v_\kappa, u, 2^u) &> 0 \quad \text{and} \\ \max\{v_0, v_1, \ldots, v_\kappa, u\} &> \mathcal{C}(a)\,. \end{aligned}$$

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Clue: Refer to an explicit enumeration $f_0, f_1, f_2, \ldots$ of all monadic partially computable functions and to a univocal representation D *à la* Matiyasevich of the relation $f_{a_1}(a_1) = a_2$, as shown above. Put:

$$H(a, x_0, x_1, \ldots, x_\kappa, y, w) =_{\text{Def}} 1 - D^2(a, x_0, x_1, \ldots, x_\kappa, y, w)\,.$$

Consider the increasing sequence

$$\langle \boldsymbol{y}_i \rangle_{i \in \mathbb{N}} = \langle 0,\ 1,\ 4,\ 15,\ 56 \ldots \rangle$$

of all solutions to the equation

$$3\,y^2 + 1 \quad = \quad \square\ .$$

Put:

$$\mathcal{D}(p, q) \quad \Longleftrightarrow_{\mathsf{Def}} \quad \exists\, \ell\ \exists\, x \left[\, q = \boldsymbol{y}_{2^{2\,\ell+1}} \ \ \&\ \ q \ = \ (2\,x + 1)\,p \,\right].$$

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Then it turns out that $\mathcal{D}$ satisfies the above-stated condition and Julia Robinson's *exponential-growth* properties

$\mathcal{D}(p,q)$ implies $q < p^p$,
for each $k \geqslant 0$ there are p and q s.t. $\mathcal{D}(p,q)$ & $p^k < q$.

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Also, $\mathcal{D}$ admits a finite-fold Diophantine repr. ***if*** the equation

$$3 \cdot \left(r^2 + 3\,s^2\right)^2 - \left(u^2 + 3\,v^2\right)^2 \quad = \quad 2$$

admits at most finitely many solutions in $\mathbb{N}$.

A Prospected $\mathcal{D}$ (Omodeo–Cantone, 2017)

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